lecture No. 19

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Measure and Integration

$f_n \in \mathbb{L}^+, \; n \geq 1$

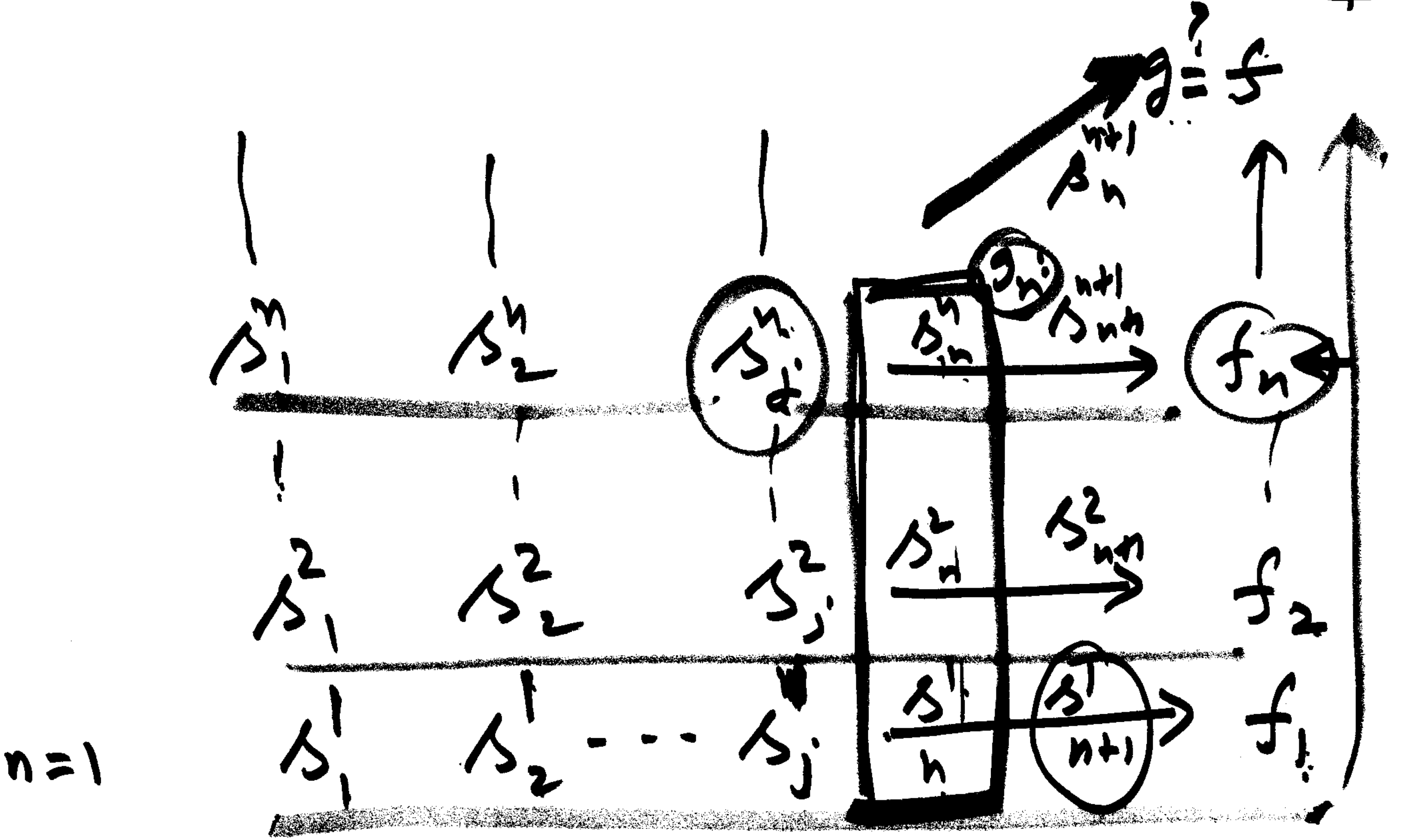
$\Rightarrow \exists \{s_j^n\}_{n \geq 1}$ such that

$s_j^n \in \mathbb{L}_0^+ \;\; \forall \; n, j$

$s_j^n \longrightarrow f_n$ as $j \to \infty$

$f_n \uparrow f$

$\Rightarrow f \in \mathbb{L}^+, \; \int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$


$g \stackrel{?}{=} f$
s_n^{n+1}
g_n
s_1^n s_2^n s_j^n s_n^n s_{n+1}^{n+1} f_n
s_1^2 s_2^2 s_j^2 s_n^2 s_{n+1}^2 f_2
$n=1$ s_1^1 s_2^1 ... s_j^1 s_n^1 s_{n+1}^1 f_1

Define

$$g_n = \max\{ S_j^y \mid 1 \le j \le n \}$$

$$g_n = \max\{ \beta_n^j \mid 1 \leq j \leq n \}$$

<u>Observe</u>

$$g_n \in \mathbb{L}_0^+ \quad \forall\, n$$

g_n is increasing

let $\quad g = \lim\limits_{n \to \infty} g_n$

Then, $\; g \in \mathbb{L}^+$

Also $\quad g_n \leq f_n \;\; \forall\, n$

$$\Rightarrow \quad g_n \le f_n \le f \quad \forall n$$

$$\Rightarrow \quad g \le f \quad \text{———} \quad (1)$$

Claim $f \le g$.

Note $\forall\ 1 \le j \le n$

$$s_n^j \le g_n \quad \forall \mapsto$$

~~Fix~~ $s_n^j \le g_n \le g \quad \forall \mapsto$

Fix j and let $n \to \infty$

$$s_n^j \longrightarrow f_j \quad \mapsto$$

$$\Rightarrow \quad f_j \leq g \quad \forall j$$

$$\Rightarrow \quad f \leq g \quad \text{——— (2)}$$

$$\Rightarrow \quad f = g$$

Hence, $f \in \mathcal{L}^+$.

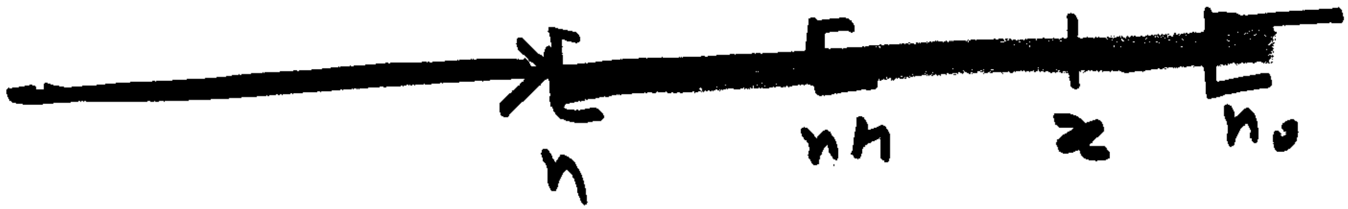
Note

$$\int f \, d\mu = \int g \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu$$

$$\lim_{n \to \infty} \int f_n \, d\mu \leq \int f \, d\mu$$

$$\Rightarrow \quad \int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu. \quad \square$$

$$\xrightarrow{\quad f_n \quad}$$



$$f_n(x) \geq f_{n+1}(x) \quad \forall \; x$$

$$f_n \in \mathbb{L}^+, \; f_n \downarrow f(x) = 0 \quad \forall x$$

$$\forall \; x \in \mathbb{R}, \; \exists \; n_0(x) \text{ s.t. } \; n_0(x) > x$$

$$\Longrightarrow f_n(x) \chi_{[n_0, \infty)}(x) = 0 \quad \forall \; n \geq n_0$$

$$\Longrightarrow \; f_n(x) \longrightarrow f(x) = 0.$$

$$\int f_n \, d\lambda = \int \chi_{[n,+\infty)} \, d\lambda$$

$$= \lambda([n,+\infty))$$

$$= +\infty \quad \forall n$$

$$\int f \, d\lambda = 0$$

$$\Longrightarrow \int f_n \, d\lambda \not\longrightarrow \int f \, d\lambda$$

$\{f_n\}_{n \geq 1}$, $f_n \in \mathbb{L}^+$

$$\left(\liminf_{n \to \infty} f_n\right)(x) = \sup_m \left[\inf_{m \geq n} \{ f_m(x) \}\right]$$

$$\phi_n(x) = \inf_{m \geq n} \{f_m(x)\}, \quad x \in X.$$

<u>Note</u> $\phi_n \in \mathbb{L}^+$

ϕ_n is increasing.

$$\left(\phi_{n+1}(x) \geq \phi_n(x) \ \forall n \right)$$

and $$\lim_{n\to\infty} \phi_n = \liminf_{n\to\infty} f_n$$

$\Longrightarrow$ (Monotone convergence Thm)

$$\int\left(\lim_{n\to\infty} \phi_n \, d\mu\right) = \lim_{n\to\infty} \int \phi_n \, d\mu$$

$$\|$$

$$\underline{\int\left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu}$$

$$\left[\ \phi_n \leq f_n \Longrightarrow \lim_{n\to\infty} \int \phi_n \, d\mu \leq \liminf \int f_n \, d\mu \ \right]$$

$\square$

$(X, \mathcal{S}, \mu)$

$f : X \longrightarrow \mathbb{R}^*$

To define $\int f \, d\mu = ?$

<u>Recall</u> $\boxed{f = f^+ - f^-}$

where $f^+, f^- \geq 0$

If f is measurable $\Longleftrightarrow f^+, f^-$ are measurable

$$\int f \, d\mu = \underline{\int f^+ d\mu} - \underline{\int f^- d\mu} \;?$$

It is defined if

$$\int f^+ d\mu < +\infty$$

and $$\int f^- d\mu < +\infty$$

$f, g \in \mathbb{L}^{+}$

$\underline{|f(x)| \leq g(x) \text{ a.e.}(\mu)}$

Claim $\quad g \in L_1(X) \Longrightarrow f \in L_1(X)$.

Note $\quad N = \{ x \in X \mid |f(x)| \not\leq g(x) \}$

Then $\quad N \in \mathcal{S}, \ \mu(N) = 0.$

Note $\quad |f| \in \mathbb{L}^{+}, \ g \in \mathbb{L}^{+}$

$|f(x)| \leq g(x)$ on N^c

$$\Longrightarrow \int |f(x)| \, d\mu(x) = \int_N |f(x)| \, d\mu(x) + \int_{N^c} |f(x)| \, d\mu$$

$$= 0 + \int_{N^c} |f(x)|\,d\mu(x)$$

$$\leq \int_{N^c} g(x)\,d\mu(x)$$

$$\leq \int g(x)\,d\mu(x) < +\infty.$$

Hence $\int |f|\,d\mu < +\infty$

Note $f^+ \leq |f|$, $f^- \leq |f|$

$\Rightarrow$ $\int f^+ d\mu$, $\int f^- d\mu \leq \int |f| d\mu < +\infty$

$\Longrightarrow f \in L_1$

Further $|f| < g$

$$\Rightarrow \int |f| \, d\mu = \cancel{\int f^+ d\mu + \int f^- d\mu}$$

$$\leq \int g \, d\mu$$

$$f(x) = g(x) \text{ a.e. } (\mu)$$

$$N = \{x \in X \mid f(x) \neq g(x)\}$$

$$\mu(N) = 0.$$

$$f(x) = g(x) \quad \forall \, x \in N^c$$

$$f \in L_1 \Rightarrow g \in L_1 ?$$

$$f(x) = g(x) \text{ a.e.}$$

$$\Rightarrow |f(x)| = |g(x)| \text{ a.e.}$$

$$\Rightarrow \underline{\int |f| \, d\mu} = \int |g| \, d\mu$$

$$\Rightarrow \int |g| \, d\mu < +\infty \Rightarrow g \in L_1$$

$$\int g \, d\mu = \int g^+ d\mu - \int g^- d\mu$$

$$= \int f^+ d\mu - \int f^- d\mu$$

$$= \int f \, d\mu.$$

$$f^+ \leqslant |f|, \quad f^- \leqslant |f|$$

$$\Rightarrow \quad \int f^+ d\mu < \int |f| \, d\mu < +\infty$$

$$\int f^- d\mu < \int |f| \, d\mu < +\infty$$

$$\Rightarrow \quad f \in L_1$$

$$f \in L_1(X) \iff |f| \in L_1(X)$$

$$\Downarrow$$

$$\int f^+ d\mu < +\infty, \quad \int f^- d\mu < +\infty$$

$$\Downarrow \quad |f| = f^+ + f^-$$

$$\underbrace{\int f^+ d\mu + \int f^- d\mu}_{\displaystyle \int |f| d\mu} < +\infty$$

$$\int |f| d\mu < +\infty$$

Conversely $\int |f| d\mu < +\infty$

$$a \in \mathbb{R}, \quad f \in L_1$$

$$|af| \leq |a|\,|f|$$

$$\int |af| \leq |a| \int |f|\, d\mu < +\infty$$

$$\Longrightarrow \quad af \in L_1$$

$$\int (af)\, d\mu = \int (af)^+ d\mu - \int (af)^- d\mu$$

$$\overset{a>0}{=} \int a\, f^+ d\mu - \int a f^- d\mu$$

$$< +\infty$$